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EVALUATION OF A DEFINITE INTEGRAL.*

BY A. PELL.

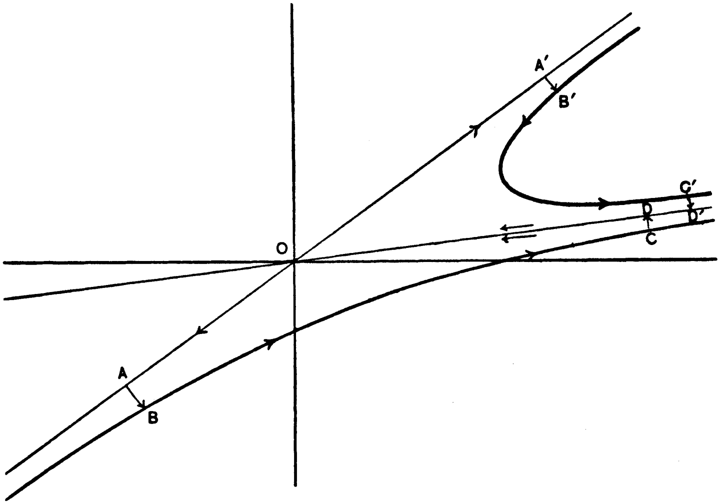
IN his book on definite integrals, Bierens de Haan gives the following formulæ:

$$\int_0^\infty e^{-(px^2+q/x^2)} \begin{matrix}\sin\\ \cos\end{matrix} \left\{ rx^2 + \frac{s}{x^2} \right\} dx = \frac{\sqrt{\pi}}{2a}\, e^{-2ab\cos(\alpha+\beta)} \begin{matrix}\sin\\ \cos\end{matrix} \left\{ 2ab\sin(\alpha+\beta)+\alpha \right\},$$

where:

$$a^4 = p^2 + r^2, \qquad b^4 = q^2 + s^2, \qquad \alpha = \tfrac{1}{2}\tan^{-1}\frac{r}{p}, \qquad \beta = \tfrac{1}{2}\tan^{-1}\frac{s}{q}.$$

For the proof reference is made to a book by Helmling,† which seems to be quite inaccessible, as it is to be found neither in the Göttingen nor in the Berlin university library.



We will establish the above formulæ on the supposition that $p > 0$, $q > 0$,‡ by making use of the fact that the integral of e^{-z^2} around any closed contour in the complex z-plane is zero. The contour we shall use is the one marked in the figure $OABCDO$, and consists of an arc of a hyperbola BC, of portions OA and DO of its asymptotes, and of the arcs AB and CD of two circles with centre at O. Letting $z = X + iY$ we will represent our hyperbola by the two equations:

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* This paper was read before the Chicago section of the American Mathematical Society at the Meeting of Dec. 28, 1899.

† *Transformation und Ausmittelung bestimmter Integrale*, 1854.

‡ If either p or q is negative the above integral has no meaning. We will not consider cases in which p or q is zero.

$$X = Px - \frac{Q}{x}, \qquad Y = Rx - \frac{S}{x},$$

where x is a parameter, and P, Q are both positive. Eliminating x between these two equations, we get as the equation of the hyperbola:

$$SRX^2 + PQY^2 - (PS + RQ)XY - (PS - RQ)^2 = 0.$$

Integrating around the contour we have:

$$\int_{OABCDO} e^{-z^2}\,dz = 0 = \int_{OA} + \int_{AB} + \int_{BC} + \int_{CD} + \int_{DO}$$

When the radii of the arcs AB and CD are increased indefinitely:*

$$\lim \int_{CD} = \lim \int_{AB} = 0, \qquad \text{and} \qquad \lim \int_{OA} = \lim \int_{DO} = -\frac{\sqrt{\pi}}{2},$$

provided that the slopes of the asymptotes are numerically less than 1; *i. e.* provided $S^2 < Q^2$ and $R^2 < P^2$. We will henceforth assume that these inequalities hold, so that we have:

$$\lim \int_{BC} e^{-z^2}\,dz = \sqrt{\pi}.$$

When z varies along BC, the parameter x varies from 0 to ∞, hence substituting and carrying out the indicated operations, we find:

$$\int_0^\infty e^{-[(P^2-R^2)x^2+(Q^2-S^2)/x^2]} \left\{ \cos 2\left[RPx^2 + \frac{SQ}{x^2} - (RQ + SP)\right] - i \sin 2 \left[RPx^2 + \frac{SQ}{x^2} - (RQ + SP)\right] \right\} \left(P + \frac{Q}{x^2} + i\left(R + \frac{S}{x^2}\right)\right) dx = e^{-2(PQ-RS)}\sqrt{\pi}.$$

Separating real and imaginary parts of this equation we shall have:

$$\int_0^\infty e^{-[(P^2-R^2)x^2+(Q^2-S^2)/x^2]} \left[\left(P + \frac{Q}{x^2}\right) \cos 2 \left\{ RPx^2 + \frac{SQ}{x^2} - (RQ + SP) \right\} + \left(R + \frac{S}{x^2}\right) \sin 2 \left\{ RPx^2 + \frac{SQ}{x^2} - (RQ + SP) \right\} \right] dx = e^{-2(PQ-RS)} \cdot \sqrt{\pi},$$

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* Picard, *Traité d'Analyse*, Vol. II, p. 159.

$$\int_0^x e^{-[(P^2-R^2)x^2+(Q^2-S^2)/x^2]}\Big[(P+\frac{Q}{x^2})\sin 2\Big\{RPx^2+\frac{SQ}{x^2}-(RQ+SP)\Big\}$$
$$-(R+\frac{S}{x^2})\cos 2\Big\{RPx^2+\frac{SQ}{x^2}-(RQ+SP)\Big\}\Big]dx=0.$$

Integrating the same function along the conjugate hyperbola:

$$X=Px+\frac{Q}{x}, \qquad Y=Rx+\frac{S}{x},$$

we find two more equations:

$$\int_0^\infty e^{-[(P^2-R^2)x^2+(Q^2-S^2)/x^2]}\Big[(P-\frac{Q}{x^2})\begin{matrix}\cos\\ \sin\end{matrix}\,2\Big\{RPx^2+\frac{SQ}{x^2}+(RQ+SP)\Big\}$$
$$\pm(R-\frac{S}{x^2})\begin{matrix}\sin\\ \cos\end{matrix}\,2\Big\{RPx^2+\frac{SQ}{x^2}+(RQ+SP)\Big\}\Big]dx=0.$$

Let α and β be the angles which the asymptotes make with the axis of reals. We have:

$$\tan\alpha=\frac{R}{P}, \qquad \tan\beta=\frac{S}{Q}\cdot$$

Let also:

$$P^2-R^2=p, \quad Q^2-S^2=q, \quad 2PR=r, \quad 2QS=s, \quad \sqrt{p^2+r^2}=a^2, \quad \sqrt{q^2+s^2}=b^2.$$

We then have:

$$\tan 2\alpha=\frac{r}{p}, \qquad \tan 2\beta=\frac{s}{q},$$

$$P=a\cos\alpha, \quad R=a\sin\alpha, \quad Q=b\cos\beta, \quad S=b\sin\beta,$$

and the only conditions to which p, q, r, s need be subjected are evidently:

$$p>0, \qquad q>0.$$

By substituting these values of P, Q, R, S in the four integrals obtained above and combining them we get:

$$\int_0^\infty e^{-(px^2+q/x^2)}\Big\{\cos\alpha\cdot\begin{matrix}\cos\\ \sin\end{matrix}\Big(rx^2+\frac{s}{x^2}\Big)\pm\sin\alpha\cdot\begin{matrix}\sin\\ \cos\end{matrix}\Big(rx^2+\frac{s}{x^2}\Big)\Big\}dx$$
$$=\frac{\sqrt{\pi}}{2a}e^{-2ab\cos(\alpha+\beta)}\cdot\begin{matrix}\cos\\ \sin\end{matrix}\Big\{2ab\sin(\alpha+\beta)\Big\},$$

from which the formulæ given at the beginning of this paper follow.

UNIVERSITY OF SOUTH DAKOTA, DECEMBER, 1899.